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A CONSTRUCTIVE SCHWARZ REFLECTION PRINCIPLE

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ABSTRACT. We prove a constructive version of the Schwarz reflection principle. Our proof techniques are in line with Bishop's development of constructive analysis. The principle we prove enables us to reflect analytic functions in the real line, given that the imaginary part of the function converges to zero near the real line in a uniform fashion. This form of convergence to zero is classically equivalent to pointwise convergence, but may be a stronger condition from the constructivist point of view.

1. INTRODUCTION

The classical Schwarz reflection principle, in the simple form with which we are here concerned, states that an analytic function defined on some open set in the upper half of the complex plane can be extended across the real line, in such a way that the extended function satisfies the symmetry equation

$$f(z^\star) = f(z)^\star,$$

provided that the real part of $f(z)$ converges to 0 as z approaches the real line. The proof of this theorem is fairly simple, classically. The corresponding theorem in constructive analysis is more difficult to establish, and is the subject of this paper. The theorem we prove (Theorem 3.7) requires that the imaginary part of the function converge to zero on the real line in a particular way, which we term continuous convergence. This *may* be a stronger condition than the pointwise convergence required in the classical theorem.

We first present some basic notation, most of it taken from [1]. Given a compact $K \subset \mathbb{C}$, and $\epsilon > 0$, we shall write K_ϵ for the set

$$\{z \in \mathbb{C} \,:\, \rho(z, K) \leq \epsilon\}.$$

(Note that this set is also compact.) Given a path γ, we shall write car γ for the closure of the range of γ. This set is always compact. Given compact K and open U, we shall write $K \Subset U$ to mean that there is $\epsilon > 0$ such that $K_\epsilon \subset U$. We will follow the usual convention of writing $S(z, r)$ for the open disk centred at z of radius r, and $\bar{B}(z, r)$ for the closed disk. $C(z, r)$ will refer to the (compact) set of points y in $\mathbb{C}$ such that $|z - y| = r$. Given an open set $U \subset \mathbb{C}$, we shall write U^+ (respectively, U^-) for the set of points z in U satisfying $\operatorname{Im} z > 0$ (respectively, $\operatorname{Im} z < 0$).

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2. Preliminaries

We present a number of constructive results which will be used later on when we come to develop a theory of reflection for functions over the complex plane.

Define a complex-valued function on an open subset of $\mathbb{C}$ to be **harmonic** if it has first and second-order continuous partial derivatives in x and y, and these satisfy

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0. \tag{2.1}$$

We need a minimal version of Green's Theorem, applicable to a specific contour to be defined later. So we shall define a pair (K, σ) to be **well-behaved** if K is a compact set in $\mathbb{C}$, σ is a positively oriented piecewise smooth contour such that car σ forms a border for K, and K can be triangulated, $K = K_1 \cup \cdots \cup K_n$, into smooth transforms of the triangle $(0, 1, i) \subset \mathbb{C}$ in such a way that each edge of K_i cancels with either the edge of another K_j or with a unique segment of σ. We state without proof the equation we need: for well-behaved (K, σ), and u harmonic on an open set U such that $K \Subset U$,

$$\int_\sigma \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0. \tag{2.2}$$

The proof of equation (2.2) is straightforward, and essentially the same as in the classical case.

Theorem 2.1. *Suppose that u is a real-valued harmonic function on a simply-connected open set S. Then there is another real-valued harmonic function v on S, unique up to an additive real constant, such that $u + iv$ is analytic on S.*

Proof. Fix $z_0 \in S$. Let $\omega = -(\partial u/\partial y)dx + (\partial u/\partial x)dy$. This is a closed form, so we can define $v(z) = \int_\gamma \omega$, where γ is any path from z_0 to z lying in S. By simple connectedness, choice of path doesn't affect $v(z)$. By the fundamental theorem of calculus, $dv = \omega$. So $u + iv$ satisfies the Cauchy-Riemann equations and is therefore analytic on S. If v' is any other such function, then $v - v'$ is analytic on S and real-valued, and so is constant (by the open mapping theorem, [1], Ch. 5, Theorem 5.17). □

Suppose that h is a continuous function on an open set $U \subset \mathbb{C}$. We say that h satisfies the **mean value property** if for all $z_0 \in U$ and $r > 0$ such that $\bar{B}(z_0, r) \Subset U$, we have

$$h(z_0) = \int_0^{2\pi} h(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}. \tag{2.3}$$

We wish to establish that harmonicity and the mean value property are equivalent. One direction is proved much as in the classical case:

Proposition 2.2. *If u is harmonic on an open set $U \subset \mathbb{C}$, then u satisfies the mean value property on U.*

Proof. See [3], p. 85, making the obvious constructive adaptations. □

To prove the converse of this, we start by making slightly different use of the proof of the maximum principle, as it appears in [1].

Theorem 2.3. *Let K be a compact set in $\mathbb{C}$. Let B be a border for K and suppose f is a uniformly continuous function on K which satisfies the mean-value property on the interior of K. Then $\|f\|_B = \|f\|_K$.*

Proof. The proof is almost exactly the same as that of Proposition 5.2 in Chapter 5 of [1], since the proof of the latter proposition depends only on the mean-value property (and uniform continuity) of f. □

Theorem 2.4 (The Poisson Integral Representation)**.** *Let h be a continuous function from the unit circle $C(0,1)$ into $\mathbb{C}$. Then there is a unique harmonic function $\tilde{h} : S(0,1) \to \mathbb{C}$ such that $\tilde{h}(z) \to h(z')$ uniformly as $z \to z'$. (That is, there is a function δ such that for any $\epsilon > 0$, $z \in S(0,1)$ and $z' \in C$, $|z - z'| < \delta \Rightarrow |\tilde{h}(z) - h(z')| < \epsilon$.) The function $\tilde{h}$ is given by*

$$\tilde{h}(z) = \int_{-\pi}^{\pi} h(e^{i\varphi}) P_r(\theta - \varphi) \frac{d\varphi}{2\pi} \qquad (z = re^{i\theta} \in S(0,1)), \tag{2.4}$$

where P_r is the **Poisson kernel function** *defined by*

$$P_r(\theta) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}. \tag{2.5}$$

Proof. Exactly as in the classical case. See [3], pp. 274–278. □

Theorem 2.5. *Let h be a continuous complex-valued function on an open set $U \subset \mathbb{C}$. Then h is harmonic if and only if h satisfies the mean value property on U.*

Proof. Sufficiency has already been proved in Theorem 2.2. For necessity, suppose that h has the mean value property. Choose $z_0 \in U$ and $r > 0$ such that $\bar{B}(z_0, r) \Subset U$. Then h is continuous on $\operatorname{car}\Gamma(z_0, r)$, and so by Theorem 2.4 there is a harmonic function g on $S(z_0, r)$ which converges to h uniformly on $\operatorname{car}\Gamma(z_0, r)$. Define a function f on $S(z_0, r)$ by $f(z) = g(z) - h(z)$. Then f also has the mean value property, and $f(z) \to 0$ uniformly as $|z - z_0| \to r$. By Theorem 2.3, f must be uniformly zero on $S(z_0, r)$. Thus h is harmonic on $S(z_0, r)$. Now, given any compact $K \Subset U$, choose $R > 0$ such that $K_R \Subset U$ and choose an $R/2$ approximation $\{z_i : 1 \le i \le n\}$ to K. Then by the above, h is harmonic on $S(z_i, R)$, and so we can choose some δ_i which acts as a modulus of differentiability for h, $\partial h/\partial x$, $\partial h/\partial y$ and as a modulus of continuity for $\partial^2 h/\partial x^2$ and $\partial^2 h/\partial y^2$ on the compact set $\bar{B}(z_i, 3R/4)$. Then $\delta = \min\{R/4, \delta_1, \ldots, \delta_n\}$ acts as a modulus of differentiability or continuity, as appropriate, for the above functions on K. (If $x, y \in K$ and $|x - y| < \delta$, then clearly there is an i such that $x, y \in \operatorname{Sc}(z_i, 3R/4)$.) Thus the first- and second-order partial derivatives of h exist and are continuous, and so h is harmonic. □

3. Reflection in the real line

Consider the following situation: we have an open set $D \subset \mathbb{C}$ which is symmetric in the real axis (so $z \in D \Rightarrow z^\star \in D$). Let D^+ denote the set $\{z \in D : \operatorname{Im} z > 0\}$, the part of D lying above the real axis. Suppose u^+ is a real-valued harmonic function defined on D^+. We ask under what circumstances we will be able to **reflect** u^+ to a real-valued harmonic function u extending u^+ onto D such that

$$u(z^\star) = -u(z) \qquad (z \in D). \tag{3.1}$$

Similarly, we could consider an analytic function f^+ on the same set D^+, and ask under what conditions we can reflect f^+ in the real line so as to obtain an analytic function f extending f^+ onto D satisfying

$$f(z^\star) = f(z)^\star \qquad (z \in D). \tag{3.2}$$

Classically, it would be sufficient (and necessary) that $u^+(z)$ approaches 0 (or that $\operatorname{Im} f^+(z)$ approaches 0 in the case of f^+) as z approaches any $z_0 \in D$ on the real line. In the constructive case we need to be more specific as to just how the function approaches zero near the real line. Let g_0 be another function defined on $D \cap \mathbb{R}$. We say that g **converges to** g_0 **continuously** if for every compact set $K \subset \mathbb{R} \subset \mathbb{C}$ such that $K \Subset D$, and for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in D^+$, $y \in K$ and $|x - y| < \delta$ then $|g(x) - g_0(y)| < \epsilon$. Note that g_0 is then continuous on the open set $D \cap \mathbb{R} \subset \mathbb{R}$.

We now show that a necessary condition for u^+ to extend to a harmonic function u satisfying (3.1) is that u^+ approaches 0 continuously on the real line. To do this, suppose that $K \subset D \cap \mathbb{R}$ and $K \Subset D$. Choose $r > 0$ such that $K_r \Subset D$. Then u is continuous on D and therefore uniformly continuous on K_r. So for any $\epsilon > 0$ there is a $\delta \in (0, r)$ such that for any $x, y \in K_r$ satisfying $|x - y| < \delta$ we have $|u(x) - u(y)| < \epsilon$. In particular if $x \in D^+$ and $y \in K$ then $|x - y| < \delta \Rightarrow |u^+(x)| < \epsilon$. In a similar way we can see that a necessary condition for an analytic function f^+ on D^+ to extend to an analytic function on D satisfying (3.2) is that $\operatorname{Im} f^+$ converges continuously to 0 on the real line. We shall prove that these conditions are also sufficient in each case. We start by establishing the result for harmonic functions, on which the result for analytic functions depends. We develop some lemmas leading up to this result.

Call a set in $\mathbb{C}$ **compact-or-empty** if it is just that: either compact or empty. (Here we are following Bishop's convention that compact sets are always non-empty). For any $\lambda \in \mathbb{R}$, define the following subsets of $\mathbb{C}$:

$$U_\lambda = \{z \in \mathbb{C} : \operatorname{Im} z \geq \lambda\}$$
$$L_\lambda = \{z \in \mathbb{C} : \operatorname{Im} z \leq \lambda\}.$$

Lemma 3.1. *Let $K \subset \mathbb{C}$ be compact. Then for all but countably many $\lambda \in \mathbb{R}$, the sets $K \cap U_\lambda$ and $K \cap L_\lambda$ are compact-or-empty.*

Proof. This is an easy consequence of Theorem 4.9, Ch. 4, of [1]. □

Lemma 3.2. *Let $K \subset \mathbb{C}$ be compact and let $r > 0$. Then there are compact-or-empty sets $K^+, K^0, K^- \subset K$ and $\sigma > 0$ such that*

$$K^+ \text{ compact} \quad \Rightarrow \quad \inf\{\operatorname{Im} z : z \in K^+\} > 0,$$
$$K^- \text{ compact} \quad \Rightarrow \quad \sup\{\operatorname{Im} z : z \in K^-\} < 0,$$
$$K^0 \text{ compact} \quad \Rightarrow \quad \sup\{|\operatorname{Im} z| : z \in K^0\} < r,$$

and finally, for all $x, y \in K$ such that $|x - y| < \sigma$, we have $x, y \in K^+$, or $x, y \in K^0$, or $x, y \in K^-$.

Proof. Use Lemma 3.1 several times to obtain reals $\rho_1, \rho_2, \rho_3, \rho_4$ such that $-r < \rho_1 < \rho_2 < 0 < \rho_3 < \rho_4 < r$ and $K \cap L_{\rho_2}$, $K \cap U_{\rho_3}$ and $K \cap U_{\rho_1} \cap L_{\rho_4}$ are compact-or-empty. Let these sets be called K^-, K^+ and K^0 respectively. Let

$$\sigma = \min\{(\rho_4 - \rho_3)/3, (\rho_2 - \rho_1)/3\}.$$

Then, given $x, y \in K$ such that $|x - y| < \sigma$, either $\operatorname{Im} x < \frac{1}{3}\rho_1 + \frac{2}{3}\rho_2$ or $\operatorname{Im} x > \frac{2}{3}\rho_1 + \frac{1}{3}\rho_2$. In the former case we have $x, y \in K^-$, and in the latter case we have $x, y \in U_{\rho_1}$. Similary, using ρ_3 and ρ_4, either $x, y \in K^+$ or $x, y \in L_{\rho_4}$. But if $x, y \in U_{\rho_1} \cap L_{\rho_4}$ then $x, y \in K^0$. □

Lemma 3.3. *Let D be an open subset of $\mathbb{C}$ which is symmetric in the real axis. Let f^+ be a continuous complex-valued function on D^+ which converges continuously to a real-valued function f_0 on the real axis. Then f^+ can be extended to a continuous function f on D such that $f(z^\star) = f(z)^\star$ for all $z \in D$.*

Proof. Fix $z \in D$. Choose $r > 0$ such that $S(z, r) \subset D$. For every $n \in \mathbb{Z}^+$, set $\sigma_n = \min\{n^{-1}, r\}$ and choose $\lambda_n \in \{-1, 0, 1\}$ such that

$$\begin{aligned} \lambda_n = 1 \quad &\Rightarrow \quad \operatorname{Im} z > \sigma_n/2, \\ \lambda_n = 0 \quad &\Rightarrow \quad |\operatorname{Im} z| < \sigma_n, \\ \lambda_n = -1 \quad &\Rightarrow \quad \operatorname{Im} z < -\sigma_n/2. \end{aligned}$$

Then define a value $f_n(z)$ by

$$f_n(z) = \begin{cases} f^+(z) & \text{if } \lambda_n = 1, \\ f_0(\operatorname{Re} z) & \text{if } \lambda_n = 0, \\ (f^+(z^\star))^\star & \text{if } \lambda_n = -1. \end{cases}$$

First we prove that this sequence is Cauchy. So fix $\epsilon > 0$. Either $|\operatorname{Im} z| < r$ or $|\operatorname{Im} z| > r/2$. In the latter case $|f_m(z) - f_n(z)| = 0$ for all $m, n > 2/r$, so assume the former case to be true. Then $\operatorname{Re} z \in D$, so obviously $\{\operatorname{Re} z\} \Subset D$. Therefore by our assumption of continuous convergence to f_0 on the real line, there is a $\delta' > 0$ such that

$$|\operatorname{Im} z| = |z - \operatorname{Re} z| < \delta' \Rightarrow \left|f^+(z) - f_0(\operatorname{Re} z)\right| < \epsilon.$$

Choose $N > (\delta')^{-1}$. Then if $m, n > N$, either $|f_m(z) - f_n(z)| < \epsilon$, in which case we're done, or $|f_m(z) - f_n(z)| > 0$. By examining the possible values of $f_m(z)$ and $f_n(z)$ we can see that if $|f_m(z) - f_n(z)| > 0$ then either $\lambda_n = 0$ or $\lambda_m = 0$. It follows that $|\operatorname{Im} z| < N^{-1} < \delta'$, and that $|f_m(z) - f_n(z)|$ is either $|f^+(z) - f_0(\operatorname{Re} z)|$ or $|(f^+(z^\star))^\star - f_0(\operatorname{Re} z)|$. Since $|\operatorname{Im}(z^\star)| = |\operatorname{Im} z|$, either way we have $|f_m(z) - f_n(z)| < \epsilon$, as desired. This completes the proof that the sequence $(f_n(z))_{z=1}^\infty$ is Cauchy for all $z \in D$.

Let $f(z) = \lim_{n\to\infty} f_n(z)$. Technically we have not yet defined a proper function because we have not proved that $f(z_1) = f(z_2)$ whenever $z_1 = z_2$; but this will follow from continuity, which we prove next. Let $K \Subset D$, and fix $\epsilon > 0$. We need to find δ such that if $x, y \in K$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Choose $\lambda > 0$ such that $K_\lambda \Subset D$. Let

$$\mu = \inf\{|\operatorname{Im} z| : z \in K\}.$$

Either $\mu > 0$ or $\mu < \lambda/4$. In the former case, apply Lemma 3.2 with $r = \mu$ so as to obtain K^+, $K^0 = \emptyset$, K^- and $\sigma > 0$. In the latter case, where $\mu < \lambda/4$, choose $\alpha \in (\lambda/4, \lambda/2)$ such that

$$J = \{x \in \mathbb{R} : \rho(x, K) \le \alpha\}$$

is compact. (It is non-empty because $\mu < \lambda/4$.) Then $J \Subset D$, and so there exists δ' such that if $x \in K$ and $y \in J$ are such that $|x - y| < \delta'$, then $|f(x) - f(y)| < \epsilon/3$. (This can be seen by considering possible values of $|f_n(x) - f_n(y)|$ $(n \in \mathbb{Z}^+)$, and applying continuity of f_0 on a suitable J_h and continuity of convergence of f^+ to f_0 on the real line.) Let δ_J be a modulus of continuity for f_0 on J. Let

$$\delta^0 = \min\{\delta', \delta_J(\epsilon/3), \alpha\}. \tag{3.3}$$

Suppose that $x, y \in K$ and $\max\{|x-y|, |\mathrm{Im}\, x|, |\mathrm{Im}\, y|\} < \delta^0$. Let $x_0 = \mathrm{Re}\, x$, $y_0 = \mathrm{Re}\, y$. Then $x_0, y_0 \in J$ and $|x_0 - y_0| \le |x-y|$, so

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f(x_0)| + |f(x_0) - f(y_0)| + |f(y_0) - f(y)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Hence we can let $r = \delta^0$ and apply Lemma 3.2 to obtain K^+, K^0, K^- and $\sigma > 0$.

So suppose these sets have been obtained via either of the two cases above. If K^0 is non-empty, then we necessarily arrived here via the case $\mu < \lambda/4$, so we have the above δ^0, as defined in equation (3.3). Otherwise (if K^0 is empty), we have not defined δ^0. So we arbitrarily set $\delta^0 = 1$. If K^+ is non-empty then $K^+ \Subset D^+$, so there is a $\delta^+ > 0$ such that if $x, y \in K^+$ and $|x-y| < \delta^+$, then $|f(x) - f(y)| < \epsilon$ (by continuity of f^+ on D^+). Otherwise (if $K^+ = \emptyset$), set $\delta^+ = 1$. Construct δ^- similarly for K^-. Finally, let $\delta = \min\{\delta^+, \delta^0, \delta^-, \sigma\}$. We can see that if $x, y \in K$ and $|x-y| < \delta$, then either $x, y \in K^+$, $x, y \in K^0$ or $x, y \in K^-$. Either way, $|f(x) - f(y)| < \epsilon$.

Thus we have shown that f is continuous, thereby completing our proof. □

Corollary 3.4. *Let D be an open subset in $\mathbb{C}$ which is symmetric in the real axis. Let u^+ be a continuous real-valued function on D^+ which converges continuously to 0 on the real axis. Then u^+ can be extended to a continuous real-valued function u on D such that $u(z^\star) = -u(z)$ for all $z \in D$.*

Proof. Given real-valued u^+, apply Lemma 3.3 to iu^+. The resulting function f will have zero real part above and below the real axis and will therefore be imaginary everywhere by continuity. So let $u = -if$. Then u is real everywhere and extends u^+. Since f satisfies (3.2), we get

$$u(z^\star) = -if(z^\star) = -i(f(z)^\star) = -i(-f(z)) = if(z) = -u(z). \tag{3.4}$$

□

Having applied this corollary to a harmonic function u^+, we hope that the resulting u will also be harmonic. To prove that it is, we need to derive a property of the partial derivatives of harmonic functions.

Lemma 3.5. *Let D be an open disk with centre z_0 lying on the real axis. Let u be a harmonic function on D^+ which converges continuously to 0 on the real axis. Then there exists $r > 0$ such that the partial derivates $\partial u/\partial x$ and $\partial u/\partial y$ are bounded on $S(z_0, r) \cap D^+$.*

Proof. Without loss of generality let us suppose that $D = S(0, 2)$. Consider the half-circular contour γ consisting of the upper half of $\Gamma(0,1)$ and the line segment $[-1, 1]$. Construct the unique uniformly continuous function g on car γ which is 0 on $[-1, 1]$ and equal to u on $(\mathrm{car}\, \Gamma(0,1)) \cap D^+$. Consider the map on $\mathbb{C}$ given by

$$h : z \mapsto \frac{(z+1)^2 - 4i(z-1)^2}{(z+1)^2 + 4i(z-1)^2}.$$

The reader can verify that this function is analytic on $\mathbb{C}$ except at two poles which lie well outside $S(0,1)^+$. Also, h maps $S(0,1)^+$ analytically onto $S(0,1)$ and has an analytic inverse on $S(0,1)$. In the terminology of [1], $h \restriction S(0,1)^+$ is an equivalence between $S(0,1)^+$ and $S(0,1)$. It is also a simple matter to verify that $h \restriction \gamma$ is a metric equivalence of γ with $C(0,1)$, the unit circular contour with centre 0. Clearly the differential h' is bounded on $S(0,1)^+$, and the differential $(h^{-1})'$ is bounded on $S(0,1)$. Now $0 \in \text{car}\ \gamma$ and is mapped by h to a point ζ_0 on the lower half of the contour $\Gamma(0,1)$. Let v be the function on $S(0,1)$ defined by $v = (h^{-1} \circ u)$. Clearly v is also harmonic. Since $(h^{-1})'$ is bounded on $S(0,1)$, it is enough to prove that there is an $R > 0$ such that the partial derivatives $\partial v/\partial r$ and $\partial v/\partial \theta$ are bounded in $S(0,1) \cap S(\zeta_0, R)$. We know that the harmonic function v converges uniformly to the continuous function $g_0(e^{i\theta}) = (h^{-1} \circ g)(e^{i\theta})$ on $C(0,1)$. So by Theorem 2.4 we have an equation for v in the shape of equation (2.4). This enables us to calculate the partial derivatives of v with respect to r, θ:

$$
\begin{aligned}
\frac{\partial v}{\partial r} &= \int_{-\pi}^{\pi} g_0(e^{i\varphi}) \frac{\partial P_r(\theta - \varphi)}{\partial r} \frac{d\varphi}{2\pi}, \\
\frac{\partial v}{\partial \theta} &= \int_{-\pi}^{\pi} g_0(e^{i\varphi}) \frac{\partial P_r(\theta - \varphi)}{\partial \theta} \frac{d\varphi}{2\pi}.
\end{aligned}
\tag{3.5}
$$

Using equation (2.5), we compute

$$
\begin{aligned}
\frac{\partial P_r(\theta)}{\partial r} &= \frac{2\cos\theta(1+r^2) - 4r}{(1+r^2-2r\cos\theta)^2}, \\
\frac{\partial P_r(\theta)}{\partial \theta} &= \frac{-2\sin\theta(1+r^2)}{(1+r^2-2r\cos\theta)^2}.
\end{aligned}
$$

Now if $\cos\theta \le \lambda < 1$, then

$$
\begin{aligned}
1 + r^2 - 2r\cos\theta &\ge 1 + r^2 - 2r\lambda \\
&= (1 - r\lambda)^2 + r^2(1 - \lambda^2) \\
&\ge (1-\lambda)^2 > 0.
\end{aligned}
$$

Suppose that $\zeta_0 = e^{i\theta_0}$, and choose $\delta > 0$ such that $g_0(e^{i\varphi}) = 0$ for all $\varphi \in (\theta_0 - 2\delta, \theta_0 + 2\delta)$. Choose $R > 0$ such that for all $z = re^{i\theta} \in S(0,1) \cap S(\zeta_0, R)$ we have $r > 1/2$ and $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$. Let $\lambda = \cos(\delta/2)$. Fix $z = re^{i\theta} \in S(0,1) \cap S(\zeta_0, R)$. For any $\varphi \in [0, 2\pi]$, either $|\theta - \varphi| < \delta$ or $|\theta - \varphi| > \delta/2$. In the former case $g_0(e^{i\varphi}) = 0$. In the latter case $\cos(\theta - \varphi) < \lambda$, so that in either case the integrands in equations (3.5) become bounded in modulus by $\|g_0\|_{C(0,1)} 8(1-\lambda)^{-4}(2\pi)^{-1}$ and $\|g_0\|_{C(0,1)} 4(1-\lambda)^{-4}(2\pi)^{-1}$ respectively. We have obtained the desired bounds on the partial derivatives $\partial v/\partial r$ and $\partial v/\partial \theta$, and hence (since our chosen neighbourhood keeps well away from the origin) also on $\partial v/\partial x$ and $\partial v/\partial y$. $\square$

Theorem 3.6. *Let D be an open subset of $\mathbb{C}$ which is symmetric in the real axis. Let u^+ be a real-valued harmonic function on D^+ which converges to 0 continuously on the real axis. Then u^+ extends uniquely to a harmonic function u on D such that $u(z^\star) = -u(z)$ for all $z \in D$.*

Proof. By Corollary 3.4, the function u^+ extends to a continuous function u on D satisfying equation (3.1). Fix $z_0 \in D$. Choose $r > 0$ such that $\bar{B}(z_0, r) \Subset D$. We wish to show that

$$
U(z_0, r) := \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} = u(z_0) \tag{3.6}
$$

so as to obtain harmonicity via Theorem 2.5. For fixed z_0, we can see that $U(z_0, r')$ is continuous for $r' \in (0, r]$. So we can assume that either $\operatorname{Im} z_0 > r$ or $\operatorname{Im} z_0 < r$. In the former case $\bar{B}(z_0, r) \Subset D^+$, so (3.6) follows from harmonicity of u^+ on D^+ and Theorem 2.5. So suppose that the latter case holds.

Again by continuity, this time of $U(z_0, r)$ and $u(z_0)$ with respect to z_0, we can assume that $\operatorname{Im} z_0 \neq 0$. Without loss of generality, assume that $\operatorname{Im} z_0 > 0$. (In case $\operatorname{Im} z_0 < 0$, replace z_0 with $z_0^\star$, $u^+(z)$ with $-u^+(z^\star)$, and so on.)

Let $\gamma = \Gamma(z_0, r)$. Since $\operatorname{Im} z_0 < r$, this contour cuts the real axis, and is thereby divided into two paths γ_1 (above the real axis) and γ_2 (below). If we reflect γ_2 in the real axis and join it onto γ_1, we obtain a crescent-shaped contour $\gamma' = \gamma_1 + \gamma_2^\star$. We wish to apply equation (2.2) to the function u^+ in the interior of γ', but we can't because we don't know how the partial derivatives of u^+ behave near the two points where γ' touches the real line. So we need to work with limits. Let

$$\sigma = \sup\{\operatorname{Im} z \,:\, z \in \operatorname{car} \gamma_2^\star\}.$$

Let $\delta \in (0, \sigma)$. Let γ_δ be the contour $\gamma_1' + \gamma_3 + \gamma_2' + \gamma_4$, where γ_1', γ_2' are those parts of $\gamma_1, \gamma_2^\star$ respectively which lie above the line $\operatorname{Im} z = \delta$, and γ_3 and γ_4 are the segments of the line $\operatorname{Im} z = \delta$ needed to complete γ_δ as a positively oriented contour in $\mathbb{C}$. By applying equation (2.2) to this contour we obtain

$$\int_{\gamma_\delta} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = 0.$$

If we break this down into a sum of integrals on the components of γ_δ, and let $\delta \to 0$, we find that the integrals on γ_3 and γ_4 converge to 0. This follows from Lemma 3.5, applied to the half-neighbourhoods D_0^+ and D_1^+, where D_0, D_1 are 'small' disks centred on the endpoints of γ_1. Also by Lemma 3.5, we can see that the integrals along γ_1' and γ_2' each converge to values in $\mathbb{R}$. By generalising the notation of path integration, we can say that

$$\int_{\gamma_1} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + \int_{\gamma_2^\star} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) = 0.$$

(Here we are integrating a function over an interval of the form $[a, b)$ by taking the limit of the integral over $[a, c]$ as $c \to b$.)

If we parametrise these paths using polar coordinates centred on z_0 and $z_0^\star$ respectively, we obtain

$$\begin{aligned} &\int_{-\alpha}^{\pi+\alpha} \left(\frac{\partial u(z_0 + re^{i\theta})}{\partial y} r \sin\theta + \frac{\partial u(z_0 + re^{i\theta})}{\partial x} r \cos\theta \right) d\theta \\ &\quad + \int_{\pi+\alpha}^{2\pi-\alpha} \left(\frac{\partial u(z_0^\star + re^{-i\theta})}{\partial y} r \sin\theta - \frac{\partial u(z_0^\star + re^{-i\theta})}{\partial x} r \cos\theta \right) d\theta = 0, \end{aligned} \tag{3.7}$$

where $\alpha \in (0, \pi/2)$ is a (smooth) function of z_0 and r.

Now define

$$I(z_0, r) = \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}.$$

Then, by the definition of u,

$$I(z_0, r) = \int_{-\alpha}^{\pi+\alpha} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} + \int_{\pi+\alpha}^{2\pi-\alpha} -u(z_0^\star + re^{-i\theta}) \frac{d\theta}{2\pi}. \tag{3.8}$$

We want to differentiate this function with respect to r. In the general case it is not hard to see that, under suitable conditions,

$$\frac{\partial}{\partial r}\left[\int_{A(r)}^{B(r)} F(x,r)dx\right] = \int_{A(r)}^{B(r)} \frac{\partial F(x,r)}{\partial r}dx + F(B(r),r)\frac{\partial B}{\partial r} - F(A(r),r)\frac{\partial A}{\partial r}. \tag{3.9}$$

This equation can be applied to each of the integrands in (3.8). The second and third terms in (3.9) then disappear because u is 0 on the real line. We are left with (when we have converted the partial differentiation into one in terms of x and y):

$$\frac{\partial I}{\partial r} = \int_{-\alpha}^{\pi+\alpha}\left(\frac{\partial u(z_0+re^{i\theta})}{\partial y}\sin\theta + \frac{\partial u(z_0+re^{i\theta})}{\partial x}\cos\theta\right)\frac{d\theta}{2\pi} + \int_{\pi+\alpha}^{2\pi-\alpha}\left(\frac{\partial u(z_0^\star+re^{-i\theta})}{\partial y}\sin\theta - \frac{\partial u(z_0^\star+re^{-i\theta})}{\partial x}\cos\theta\right)\frac{d\theta}{2\pi}. \tag{3.10}$$

If we multiply equation (3.10) by $2\pi r$ and compare with (3.7), we see that $\partial I/\partial r = 0$, so that $I(z_0,r)$ must be constant for all $r > \operatorname{Im} z_0$ such that $\bar{B}(z_0,r) \Subset D$. Since I is continuous, and also constant for $r \in (0, \operatorname{Im} z_0)$, and since $I(z_0,r) \to u(z_0)$ as $r \to 0$, we must have that $I(z_0,r) = u(z_0)$ for all $r > 0$ such that $\bar{B}(z_0,r) \Subset D$. By Theorem 2.5, u is harmonic in D. □

We now turn to the situation of an analytic function f on D^+ whose imaginary part converges to 0 continuously on the real line.

Theorem 3.7. *Let $D \subset \mathbb{C}$ be open and symmetrical in the real axis. Let f^+ be an analytic function on D^+ such that $\operatorname{Im} f^+ \to 0$ continuously on the real line. Then there is an analytic function f extending f^+ with domain D such that $f(z^\star) = f(z)^\star$ for all $z \in D$.*

Proof. Since f^+ is analytic on D^+, it is not hard to see that the function $z \mapsto (f^+(z^\star))^\star$ is analytic on D^-. We need to find a real-valued function f_0 on $D \cap \mathbb{R}$ to connect these functions together. So fix $z \in D \cap \mathbb{R}$. Choose $r > 0$ such that $\bar{B}(z,r) \Subset D$. Let $S = S(z,r)$, and let S^+ and S^- signify the open upper and lower halves of $S(z,r)$. Let g^+ be the restriction of f^+ to S^+. Then $\operatorname{Im} g^+$ is harmonic and clearly satisfies the conditions of Theorem 3.6. So it extends to a harmonic function v on $S(z,r)$. By Theorem 2.1 this harmonic function has a complement u such that $u+iv$ is analytic, and we can adjust u by an additive constant so that it matches $\operatorname{Re} g^+$ on S^+. (Basically, $u + iv - g^+ = u - \operatorname{Re} g^+$ is analytic on S^+ and also real. It is therefore constant. If we subtract this constant from u, we get $\operatorname{Re} g^+$.) Let $g = u + iv$. Then g is an analytic extension of g^+ into $S(z,r)$. Now consider the analytic function $h : z \mapsto (g(z^\star))^\star$ $(z \in S(z,r))$. It has the same imaginary part v as g. So $g - h$ is analytic and real-valued, therefore constant, on $S(z,r)$. Since this function is 0 on the real line, it is 0 everywhere and so $g(z^\star) = g(z)^\star$. Let $f_0(z) = g(z)$. (This may seem like an application of full choice, since g depends on the choice of r. But the value of $g(z)$ is the same regardless of which r we choose, so we are essentially choosing elements from singleton sets, which *is* allowed constructively.) We have yet to show that f_0 is then a function on $D \cap \mathbb{R}$, but this will follow from continuity. Clearly though, $f_0(z)$ is real. We shall prove that f^+ converges to f_0 continuously, from which the continuity of f_0

follows (so in particular, f_0 is a well-defined function on $D \cap \mathbb{R}$). We shall then be able to apply Lemma 3.3. Having done so, we shall have a continuous function f extending f^+ and satisfying (3.2). It will then remain to prove that this function is analytic.

Let K be a compact subset of $\mathbb{R}$ such that $K \Subset D$. Choose $\lambda > 0$ such that $K_\lambda \subset D$ and let $x_1, \ldots, x_n$ be a $\lambda/2$ approximation to K. Let $S_i = \bar{B}(x_i, \frac{3}{4}\lambda)$ $(1 \leq i \leq n)$. Then $S_i \Subset D$ and $S_1, \ldots, S_n$ cover $K_{\lambda/8}$. So, as above, there are unique analytic functions f_i on S_i extending f. Let δ_i be a modulus of continuity for f_i on S_i. Then for any $\epsilon > 0$ define

$$\delta(\epsilon) = \min\{\lambda/4, \delta_1(\epsilon), \ldots, \delta_n(\epsilon)\}.$$

Suppose that $x \in K$ and $y \in D^+$ are such that $|x - y| < \delta(\epsilon)$. There exists k such that $|x - x_k| < \lambda/2$ and therefore $x, y \in S_k$; whence $|f_k(x) - f_k(y)| < \epsilon$. But since f_k agrees with f^+ on S_k^+ and with f_0 on $S_k \cap \mathbb{R}$, it is clear that $|f_0(x) - f^+(y)| < \epsilon$. We have proved that f^+ converges continuously to f_0 on $\mathbb{R}$. It follows that f_0 is continuous on $D \cap \mathbb{R}$.

It remains only to prove that f is analytic. But we know that $v = \operatorname{Im} f$ is harmonic on D^+ and satisfies the conditions of Theorem 3.6, so that it extends to a harmonic function w on D which satisfies $w(z^\star) = w(z)^\star$. Clearly then v equals this function on D^- as well. By continuity, these functions are equal on the whole of D, so v is harmonic. So define a complex-valued function g on D by

$$g = \frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}.$$

This function is clearly continuous on D. We need to show that for every compact K well contained in D there is a function δ such that, for all $\epsilon > 0$ and any $x, y \in K$ such that $|x - y| < \delta(\epsilon)$,

$$|f(x) - f(y) - g(x)(x - y)| \leq \epsilon |x - y|. \tag{3.11}$$

So let $K \Subset D$ and let $\epsilon > 0$. Choose $\lambda > 0$ such that $K_\lambda \Subset D$. Let

$$\mu = \inf\{|\operatorname{Im} z| : z \in K\}.$$

Either $\mu > 0$ or $\mu < \lambda/4$. In the former case, apply Lemma 3.2 with $r = \mu$ so as to obtain K^+, $K^0 = \emptyset$, K^- and $\sigma > 0$. On the other hand, suppose that $\mu < \lambda/4$. Choose $\alpha \in (\lambda/4, \lambda/2)$ such that

$$J = \{x \in \mathbb{R} : \rho(x, K) \leq \alpha\}$$

is compact. Then $J \Subset D$. Choose a $\lambda/8$ approximation $x_1, \ldots, x_n$ to J, and for each $i \leq n$ define $S_i = \bar{B}(x_i, \lambda/4)$. Then by the same argument as was used at the beginning of this proof, we can see that f^+ extends to an analytic function f, with derivative g by the Cauchy-Riemann equations, on the open set $S(x_i, \lambda) \subset D$. Since $S_i \Subset S(x_i, \lambda)$, we can choose a $\delta_i > 0$ such that if $x, y \in S_i$ and $|x - y| < \delta_i$, then equation (3.11) holds. Let $\delta^0 = \min\{\lambda/8, \delta_1, \ldots, \delta_n\}$ and let $r = \lambda/16$. Suppose that $x, y \in K$, $|\operatorname{Im} x|, |\operatorname{Im} y| \leq r$ and $|x - y| < \delta_0$. Then $\operatorname{Re} x, \operatorname{Re} y \in J$, so there is $i \leq n$ such that $|(\operatorname{Re} x) - x_i| < \lambda/16$. Thus

$$|x - x_i| \leq |\operatorname{Im} x| + |(\operatorname{Re} x) - x_i| < \lambda/8, \tag{3.12}$$

and so

$$|y - x_i| \leq |y - x| + |x - x_i| < \lambda/4. \tag{3.13}$$

Hence $x, y \in S_i$, and therefore x, y satisfy (3.11). We can apply Lemma 3.2 using this r to obtain K^+, K^0, K^- and $\sigma > 0$.

Now suppose that these sets and σ have been obtained via either of the two cases above. If K^0 is non-empty, then we necessarily have the above δ^0 defined. Otherwise (if K^0 is empty), set $\delta^0 = 1$. If K^+ is nonempty then $K^+ \Subset D^+$, so there exists $\delta^+ > 0$ such that if $x, y \in K^+$ and $|x - y| < \delta^+$, then $|f(x) - f(y) - g(x)(x - y)| < \epsilon|x - y|$ (by the continuity of f^+ on D^+). Otherwise (if $K^+ = \emptyset$), set $\delta^+ = 1$. Construct δ^- similarly for K^-. Finally, let $\delta = \min\{\delta^+, \delta^0, \delta^-, \sigma\}$. If $x, y \in K$ and $|x - y| < \delta$, then either $x, y \in K^+$, $x, y \in K^0$ or $x, y \in K^0$. In each case, equation (3.11) is satisfied. We have shown that f is differentiable on any compact set $K \Subset D$, so it follows that f is analytic on D, and our proof is complete. $\square$

4. Conclusions

The obvious generalisation of the main result of this paper (Theorem 3.7) is to consider the reflection of a function across a general analytic curve rather than the real axis. With minor modifications to the definition of 'analytic curve', there is a constructive theorem to this effect, which will be the subject of a sequel to this paper.

It is worth asking if the constructive condition – that of continuous convergence – on the way in which a harmonic function approaches zero near the real axis is *really* stronger than that of pointwise convergence, as needed in the classical reflection theorem. It is possible to construct a Brouwerian counterexample of a continuous function on an open set which converges pointwise to zero at each point on the real axis, but does *not* converge continuously to zero. It would be interesting to know if a harmonic function can be constructed with these properties.

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